PROBLEMS.

26. Proposed by J. W. WATSON, Middle Creek, Ohio.

Find the average area of all right angled triangles having a given hypotenuse.

27. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the mean area of the *dodecagonal surface* formed by joining in order the points taken at random, one in each sector of a regular dodecagon.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

12. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If the measures of curvature and tortuosity of a curve be constant at every point of a curve, the curve will be a helix traced on a cylinder.

Solution by the PROPOSER.

A helix, inclination ω, traced on a right circular cylinder, radius r, is an unicursal curve, and may be defined by the equations $x=r\cos\theta\ldots.(1)$, $y=r\sin\theta\ldots.(2)$, and $z=r\theta\tan\omega\ldots.(3)$, in which θ is the angle through which the generating line has resolved when the point has moved through a space z on the generating line. From (1), (2), and (3), we have respectively

$$\frac{dx}{d\theta}=-r\sin\theta;\ d\left(\frac{dx}{d\theta}\right)=\frac{d^2x}{d\theta^2}=-r\cos\theta\ldots.(4),$$

$$\frac{dy}{d\theta}=r\cos\theta;\ d\left(\frac{dy}{d\theta}\right)=\frac{d^2y}{d\theta^2}=-r\sin\theta\ldots.(5),$$

$$\frac{dz}{d\theta}=r\tan\omega;\ d\left(\frac{dz}{d\theta}\right)=\frac{d^2z}{d\theta^2}=0\ldots.(6).$$

$$\therefore\ \frac{ds}{d\theta}=\sqrt{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2\right]}=\frac{r}{\cos\omega}\ldots.(7).$$

Dividing (4), (5), and (6), by the square of (7),

$$\frac{d^2x}{ds^2}=\frac{-\cos\theta\cos^2\omega}{r}\ldots.(8),\ \frac{d^2y}{ds^2}=\frac{-\sin\theta\cos^2\omega}{r}\ldots.(9),\text{ and }\frac{d^2z}{ds^2}=0\ldots.(10).$$

Since the reciprocal of the radius of curvature is the *measure of the curvature* at any point of a tortuous curve, we have

$$\frac{1}{\rho}=\sqrt{\left[\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right]}=\frac{\cos^2\omega}{r}\ldots.(11),$$

which is necessarily a *constant* quantity for every point of the curve.

The formula for the *measure of the tortuosity* at any point of a tortuous curve is, regarding τ as the radius of torsion,

$$\frac{1}{\tau}=\sqrt{\left[\left(\frac{d\lambda}{ds}\right)^2+\left(\frac{d\mu}{ds}\right)^2+\left(\frac{d\nu}{ds}\right)^2\right]}\ldots.(12),$$

in which $\dfrac{d\lambda}{ds}=\dfrac{d}{ds}\left[\rho\left(\dfrac{dy}{ds}\cdot\dfrac{d^2z}{ds^2}-\dfrac{dz}{ds}\cdot\dfrac{d^2y}{ds^2}\right)\right]\ldots.(a),$

$$\frac{d\mu}{ds}=\frac{d}{ds}\left[\rho\left(\frac{dz}{ds}\cdot\frac{d^2x}{ds^2}-\frac{dx}{ds}\cdot\frac{d^2z}{ds^2}\right)\right]\ldots.(b),$$

$$\frac{d\nu}{ds}=\frac{d}{ds}\left[\rho\left(\frac{dx}{ds}\cdot\frac{d^2y}{ds^2}-\frac{dy}{ds}\cdot\frac{d^2x}{ds^2}\right)\right]\ldots.(c).$$

From (4), (5), and (6), by means of (7), we deduce

$$\frac{dx}{ds}=-\cos\omega\sin\theta\ldots.(13,\ \frac{dy}{ds}=\cos\omega\cos\theta\ldots.(14),\text{ and }\frac{dz}{ds}=\sin\omega\ \ldots(15).$$

Reducing (a), (b), and (c), by means of (7), (8), (9), (10), (13), (14), (15): and then differentiating the results, we have respectively

$$\frac{d\lambda}{ds}=\frac{\sin\omega\cos\omega\cos\theta}{r}\ldots.(16),\quad\frac{d\mu}{ds}=\frac{\sin\omega\cos\omega\cos\theta}{r}\ldots.(17),$$

and $\dfrac{d\nu}{ds}=\dfrac{d}{ds}\left[\dfrac{r\cos^3\omega(\sin^2\theta+\cos^2\theta)}{\cos^2\omega}\right]=0\ldots.(18).$

Transforming (12) by means of (16), (17), and (18),

$$\frac{1}{\tau}=\sqrt{\left[\frac{\sin^2\omega\cos^2\omega(\sin^2\theta+\cos^2\theta)}{r^2}\right]}=\frac{\sin\omega\cos\omega}{r}\ldots.(19),$$

which is also necessarily a *constant* quantity for every point of the curve.

NOTE.—Multiplying the numerator and the denominator of the right-hand member of (19), by $\cos\omega$, we have

$$\frac{1}{\tau}=\frac{\sin\omega}{\cos\omega}\times\frac{\cos\omega}{r}=\tan\omega\times\text{the curvature.}$$

If $\omega=\frac{1}{4}\pi$, the curvature and the tortuosity are necessarily equal. Had we assumed

$$x=\left(\frac{s}{\sqrt{6}}\right)\cos\left[\left(\frac{1}{\sqrt{2}}\right)\log\left(\frac{2s^2}{3c^2}\right)\right],\ y=\left(\frac{s}{\sqrt{6}}\right)\sin\left[\left(\frac{1}{\sqrt{2}}\right)\log\left(\frac{2s^2}{3c^2}\right)\right],$$

and $z=s\div\sqrt{2}$, then $1/\rho$ and $1/\tau$ would each have equaled $1/s$; that is, the curvature and the tortuosity would then have been the same for every point of the curve. Truly, the helix is a wonderful curve; it can easily be *axised* and *pitched* so as to have the same curvature and tortuosity as any given curve, while the loci of the centers of curvature and tortuosity are similar helices traceable on the same cylinder.

Also solved by *Professor G. B. M. Zerr*.